# OPTIMAL POWER ALLOCATION BASED ON SINGULAR VALUE DECOMPOSITION OF MIMO CHANNEL MATRIX

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***ABSTRACT*: One of the attractive feature of MIMO system is a spatial multiplexing gain and consequently a higher capacity performance over single-input single-output system. In spatial multiplexing data transmission is being carried out by multiple parallel channels between transmitter and receiver. Total capacity of MIMO system is given by sum of individual capacity of all parallel channels. So to maximize this total capacity one has to allocate power optimally to each channel or stream. This paper describes capacity model of MIMO system using Singular Value Decomposition (SVD) of MIMO channel matrix. Optimal power allocation algorithm widely known as water filling algorithm has been discussed with example and simulation results..***

***Keywords— Multiple Input Multiple Output (MIMO),Singular Value Decomposition (SVD),Water filling algorithm.***



## I. INTRODUCTION

As a poor performance and limited capacity of wireless channel, need raised to pack more number of bits per Hz. As a solution of this problem MIMO system has come into picture where there are multiple antennas at transmitter and receiver side. Actually poor performance of wireless channel is due to multipath fading. MIMO uses these multipath components in constructive manner to increase received SNR.MIMO system basically provides following three features:

- Beamforming - It allows antennas to direct their main beam in appropriate direction to achieve better SNR by increasing received power[6].
- Spatial diversity - Transmitted signal is coded in space as well as time domain with some redundancy to improve BER performance of system[6].
- Spatial multiplexing - Set of data streams are transmitted in parallel from different antennas and appropriate signal processing is used at receiver to separate those data streams[6].

In MIMO system each antenna operates at same band of frequencies so no additional bandwidth is required. Also sum of transmitted power from all of the antennas is always less than or equals to transmitted power for single antenna system. So MIMO system do not require additional transmission power.

## II. MIMO SYSTEM MODEL

Consider MIMO system with t transmit antennas and r receive antennas and assume block of K symbols are transmitted in T time slots. For such MIMO system System Model is described as

$$Y = H*X + N \tag{1}$$

Where

$$Y = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,T} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r,1} & y_{r,2} & \cdots & y_{r,T} \end{pmatrix} \tag{2}$$

$$N = \begin{pmatrix} n_{1,1} & n_{1,2} & \cdots & n_{1,T} \\ n_{2,1} & n_{2,2} & \cdots & n_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ n_{r,1} & n_{r,2} & \cdots & n_{r,T} \end{pmatrix} \tag{3}$$

$$X = \begin{pmatrix} x_{1,1} & x & \cdots & x_{1,T} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ x_{t,1} & n_{t,2} & \cdots & x_{t,T} \end{pmatrix} \tag{4}$$

$$H = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,t} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r,1} & h_{r,2} & \cdots & h_{r,t} \end{pmatrix} \tag{5}$$

here Y is received symbol matrix having r x T dimensions and X is transmitted STBC codeword of t x T dimensions. H represents channel matrix of Rayleugh fading coefficients and N denotes AWGN noise matrix.

- $y_{i,j}$ ( i=1,2,......r and j=1,2,...T ) is received signal at receiving antenna i in time slot j.
- $x_{i,j}$ ( i=1,2,......t and j=1,2,...T ) is transmitted signal from transmitting antenna i at time slot j.
- $n_{i,j}$ ( i=1,2,......r and j=1,2,...T ) is noise present at receiving antenna i in time slot j. $n_{i,j}$ are i.i.d complex random variables having zero mean and $N_t$ / (2 * SNR) variance per each dimension. Therefore each entry in N is independent from remaining entries[1][7].

$$\sigma = r / 2 * SNR$$
$$n_{i,j} = \text{Normal}(0,\sigma) + j*\text{Normal}(0,\sigma)$$

- $h_{i,j}$ ( i=1,2,......r and j=1,2,...t ) is Rayleigh distributed complex fading coefficients having zero mean and unity variance per each dimensions. We also assumed that channel is uncorrelated that is entries in H are independent to each other[1][7].

$$h_{i,j} = \text{Normal}(0,1) + j*\text{Normal}(0,1)$$

## III. SINGULAR VALUE DECOMPOSITION

Consider a MIMO channel matrix H described in above section with a assumption that r ≥ t then Singular Value Decomposition of matrix H is given by

$$H = U\Sigma V^H \tag{6}$$

where U refers to column matrix of t columns, V refers to row matrix of t rows and Σ refers to singular value matrix which is diagonal matrix of t dimension[2][8].

$$U = [\, U_1 \quad U_2 \cdots U_{t-1} \quad U_t \,] \tag{7}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_t \end{bmatrix} \tag{8}$$

$$V = \begin{bmatrix} V_1^H \\ V_2^H \\ \vdots \\ V_t^H \end{bmatrix} \tag{9}$$

U is r × t matrix and V is a t × t matrix such that columns of U and rows of V are orthonormal i.e.

$$\|U_i\|^2 = 1 \; and \; U_i^H U_j = 0 \; for \; i \neq j$$

$$\|V_i\|^2 = 1 \; and \; V_i^H V_j = 0 \; for \; i \neq j$$

Also U V and Σ have the following properties [8]

$$VV^H V = VV^H = I$$

$$U^H U = I$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_t \geq 0$$

So V is a unitary matrix for any r ≥ t while U is also a unitary matrix for r = t and diagonal elements of Σ are known as singular values which are non negative and in a ordered manner.

Now consider a MIMO system model described in above section and put $H = U\Sigma V^H$

$$Y = HX + N$$
$$Y = U\Sigma V^H X + N$$

Multiply both the sides by $U^H$ (beamforming at receiver)

$$U^H Y = U^H (U\Sigma V^H X + N)$$

$$\tilde{Y} = \Sigma V^H X + \tilde{N}$$

where

$$\tilde{Y} = U^H Y \; and \; \tilde{N} = U^H N$$

Now let $X = V\tilde{X}$ (precoding at transmitter)

$$\tilde{Y} = \Sigma V^H V\tilde{X} + \tilde{N}$$

$$\tilde{Y} = \Sigma\tilde{X} + \tilde{N} \tag{10}$$

Or equivalently

$$\begin{bmatrix} \widetilde{y_1} \\ \widetilde{y_2} \\ \vdots \\ \widetilde{y_t} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_t \end{bmatrix} \begin{bmatrix} \widetilde{x_1} \\ \widetilde{x_2} \\ \vdots \\ \widetilde{x_t} \end{bmatrix} + \begin{bmatrix} \widetilde{n_1} \\ \widetilde{n_2} \\ \vdots \\ \widetilde{n_t} \end{bmatrix} \tag{11}$$

So in above equation U and V are eliminated since we have performed beamforming at receiver by matrix U and precoding at transmitter by matrix V. So it requires CSI to avail at both the sides.

Now simplify above equation, then we have

$$\widetilde{y_1} = \sigma_1 \widetilde{x_1} + \widetilde{n_1}$$

$$\widetilde{y_2} = \sigma_2 \widetilde{x_2} + \widetilde{n_2}$$

$$\widetilde{y_t} = \sigma_t \widetilde{x_t} + \widetilde{n_t}$$

From above equations we can see that all the transmitted symbols appear only to their respective receive antennas and they are not interfering simultaneously at any receive antenna. So it forms a collection of t parallel channels which are decoupled to each other. Also here t symbols are parallel transmitted by MIMO channel in single time slot. It refers to spatial multiplexing in MIMO communication.

Now consider noise matrix at receiver $\tilde{N} = U^H N$

$$E\{\tilde{N}\tilde{N}^H\} = E\{U^H N U N\}$$

$$E\{\tilde{N}\tilde{N}^H\} = \sigma_N^2 I_t U^H U$$

$$E\{\tilde{N}\tilde{N}^H\} = \sigma_N^2 I_t$$

$$\sigma_N^2 = \sigma_{\tilde{N}}^2 \tag{12}$$

where $\sigma_N^2$ is noise power.

From above equation we can say that noise power before the beamforming is identical to noise power after the beamforming. In other words beamforming do not affect noise power at receiver [8].

So SNR of $i^{th}$ channel is given by

$$SNR_i = \frac{\sigma_i^2 P_i}{\sigma_N^2} \tag{13}$$

And hence channel capacity of $i^{th}$ channel is [5]

$$C_i = \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma_N^2}\right) \tag{14}$$

We have total t such independent channels and hence total capacity of a MIMO channel is

$$C = \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma_N^2}\right) \tag{15}$$

## IV. WATER FILLING ALGORITHM

This algorithm refers to allocate optimal power to each transmission stream which maximizes total capacity of MIMO channel.

Consider total transmission power P and power allocated to $i^{th}$ channel or stream is $P_i$ then sum of power allocated to all the streams must be less than or equals to P.

$$P_1 + P_2 + \ldots + P_t \leq P$$

Hence optimal power allocation reduces to constraint maximization problem given by

$$\max_{P_i}\left(\sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma_N^2}\right)\right)$$

subjected to constraint

$$\sum_{i=1}^{t} P_i = P \tag{16}$$

In other words finding optimal $P_i$ which maximizes total capacity subjected to above constraint [8]. This constraint maximization problem can be solved using langrage multiplier method.

Consider function f as

$$f = \sum_{i=1}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma_N^2}\right) + \lambda\left(P - \sum_{i=1}^{t} P_i\right)$$

where λ is langrage multiplier

Now differentiate f with respect to $P_i$ and equate to zero to find optimal $P_i$.

$$\frac{df}{dP_i} = 0$$

$$P_i = \left(\frac{1}{\lambda} - \frac{\sigma_N^2}{\sigma_i^2}\right)^+ \tag{17}$$

where

$$x^+ = \begin{Bmatrix} x & if\ x > 0 \\ 0 & if\ x < 0 \end{Bmatrix}$$

Put this $P_i$ to constraint described in equation (16) then we have equation for optimal power allocation given by

$$\sum_{i=1}^{t} \left(\frac{1}{\lambda} - \frac{\sigma_N^2}{\sigma_i^2}\right)^+ = P \tag{18}$$

For optimal power allocation we first find $\frac{1}{\lambda}$ level according to above equation then we calculate power allocated to $t^{th}$ stream i.e. $P_t$ according to equation (17). If $P_t$ is greater than zero then we will allocate power to rest of the streams according to equation (17). But if $P_t \leq 0$ then we allocate zero power to $t^{th}$ stream and repeat the same procedure for t = t – 1 until total power is allocated to all the streams [1][3].

Steps in Water filling algorithm [8]:

1. Define channel matrix H, total transmission power P and noise power N.
2. For T transmission streams or channels calculate $\frac{1}{\lambda}$ level according to equation (18).
3. Calculate power allocated to $t^{th}$ channel according to equation (17).
4. Check whether power allocated to $t^{th}$ channel $P_t$ is positive or not.
5. If $P_t$ is positive allocate powers to each channel according to equation (17).
6. If $P_t$ is not positive then assign zero power to $t^{th}$ channel and set t = t – 1 and go back to step 2.

## V. SIMULATION RESULTS

Simulation has been done on MATLAB. We have given channel matrix H, Total transmission power P and Total noise power N as an input to water filling algorithm. As a results we are getting optimal power allocation to each individual channel.

Case-1

$$H = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Total transmission power = -1.25 dB = 0.75 W
Total noise power = 3 dB = 2 W

TABLE 1. Optimal power allocation for case 1

| Optimally allocated power | |
|---|---|
| $P_1$ | 0.4325 W |
| $P_2$ | 0.3174 W |
| $P_3$ | 0 W |
| TOTAL | 0.7499 W |

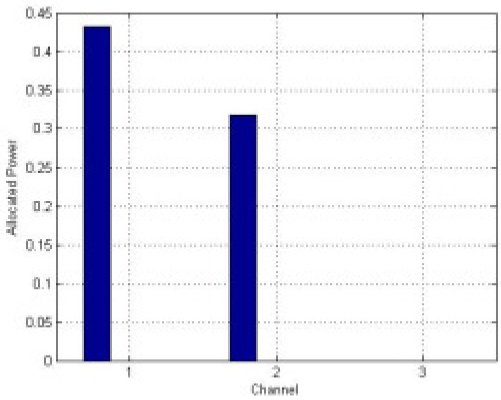


Figure.1. Optimal power allocation for case 1

Case-2

$$H = \begin{bmatrix} 2 & 6 & 3 \\ 3 & 3 & -5 \\ -3 & 5 & 2 \\ 5 & 4 & -7 \end{bmatrix}$$

Total transmission power = 0 dB = 1 W
Total noise power = 1 dB = 1.2589 W

TABLE 2. Optimal power allocation for case 2

| Optimally allocated power | |
|---|---|
| $P_1$ | 0.3723 W |
| $P_2$ | 0.3644 W |
| $P_3$ | 0.2363 W |
| TOTAL | 1 W |

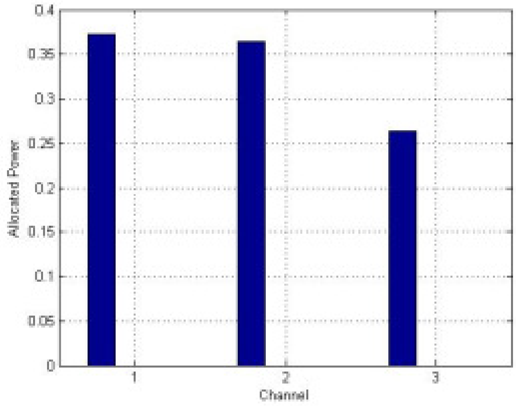


Figure.1. Optimal power allocation for case 2

## VI. CONCLUSIONS

We have discussed Singular Value Decomposition of MIMO channel matrix. Diagonal elements of singular value matrix are known as singular values and number of singular values is equals to rank of channel matrix. Rank of channel matrix determines maximum independent channels or streams that can be transmitted in parallel. We have also discussed water filling algorithm for optimal power allocation to each transmission stream. We have simulated water filling algorithm for two distinct case.

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