LDM- PADÉ TO SOLVE A RICCATI NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract
In this paper we propose an alternative method, known as The Laplace decomposition method to solve the nonlinear Riccati differential equation. We also apply the after-treatment technique known as Padé approximation to increase the accuracy of a solution obtained by the Laplace decomposition method. The numerical analysis of the presented study shows that the method is relatively easy and highly accurate.
Key words: Riccati differential equation, nonlinear ordinary differential equation, Laplace decomposition method, Padé approximation.

1 Introduction

Consider the nonlinear Riccati differential equation of the type

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \qquad (1.1)$$

$$y(x_0) = \alpha , \qquad (1.2)$$

With $P(x) \neq 0 \ \forall \ x$ and α is constant . The Riccati differential equation plays an important role in various technical applications, physical models (spring mass system, bending of beams, chemical reactions etc.) and also in Mathematical Finance . The classical approach to solve an ordinary nonlinear Riccati differential equation completely , is to use a given particular solution of (1.1) before hand and by means of the transformations, (1.1) can be reduced into a Bernoulli's type differential equation. The Bernoulli equation can be easily solved by reducing it to linear form and thus we are able to solve the Riccati differential equation completely. In the case where we don't have any particular solution available are used in a closed form then we can not use the above approach to solve the problem.
Several numerical methods are employed by researchers to solve Riccati differential equation completely . An extensive study was given by Fu [2] in his paper.
Several methods like Taylor-Maclaurin Series solution, Solution by Picard's Method, Transformation Method and Ad hoc successive approximation method have been employed by him to solve the given nonlinear Riccati differential equation . Recently Taylor Matrix method [5] and Adomian decompostition method [1] are also employed to solve the problem. For other alternative methods to solve the Riccati differential equation we refer [3],[4] and references therein.
The aim of this paper is to propose a new technique known as the Laplace-Padé method to solve the nonlinear Riccati ordinary differential equation and to compare the numerical results obtained by Euler method, Taylor matrix method and a recently popular the Adomian decomposition method . This technique includes the solution of the problem by Laplace decomposition method using the Adomian polynomials and then calculating its Padé approximation (See [6],for more details) to increase the accuracy of the solution. The results are also verified by an example and the corresponding results in Table 1 indicate the accuracy of the proposed method over the existing techniques, i.e

the Taylor –Machlaurin Series solution, Solution by Picard's Method , Transformation Method The Ad hoc successive approximation method and the Adomian Decomposition method.

2 Analysis of numerical Laplace Decomposition method

In this paper, the Laplace decomposition method is employed to obtain the solution of the following nonlinear ordinary Riccati differential equation

$$y'(x) = P(x)y^2(x) + Q(x)y(x) + R(x),$$
$$y(x_0) = \alpha, \qquad \textbf{(2.3)}$$

Where α , is a constant and $P(x) \neq 0$,for all x ,Q(x) and R(x) are given continuous functions. This mehod consists of first applying Laplace transform on both the sides of (2.3) , we have

$$\pounds(y'(x)=\pounds(P(x)\, y^2(x)) +\pounds(Q(x)y(x)) +\pounds(R(x)) \qquad \textbf{(2.4)}$$

or

$$s\pounds\,(y) -y(0) = \pounds(\, \mathrm{R}(x)\,) + \pounds(\, Q(x)y(x)\,) + \pounds(\, P(x)\, y^2(x))$$

On simplification and using the initial condition, we have

$$\pounds\,(y) = \frac{\alpha}{s} + \frac{1}{s}\,\pounds\big(\,\mathrm{R}(x)\big) + \frac{1}{s}\,\pounds\big(\,Q(x)y(x)\big) + \frac{1}{s}\pounds(\,P(x)\,y^2(x)) \qquad \textbf{(2.5)}$$

Next, the solution of the problem is taken as the sum of infinite series given as

$$y(x) = \sum_{n=o}^{\infty} y_n(x), \qquad \textbf{(2.6)}$$

Where the component $y_n(x)$ of the solution y(x) will be determined by a re-cursive formula (2.13).Also, the nonlinear term $N(y) = P(x)\, y^2$ is decomposed as follows

$$N(y) = \sum_{n=0}^{\infty} A_n \qquad \textbf{(2.7)}$$

Where Adomian polynomials A_n ,$\forall\, n \in$ N, can be calculated by the formula given as

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\,\left[\sum_{i=0}^{\infty} \lambda^i\, y_i\right]\right]_{\lambda=0} \qquad \textbf{(2.8)}$$

The few Adomian polynomials are determined as per the following scheme :

$$A_0 = f(y_{0\,)},$$
$$A_1 = f'(y_0)y_1\,,$$
$$A_2 = f'(y_0)y_2 + \frac{1}{2}\,f''(y_0)y_1^2$$
$$A_3 = f'(y_0)y_3 + \frac{1}{3}\,f'''(y_0)y_1^3,$$
$$A_4 = f'(y_0)y_4 + \left(\frac{y_2^2}{2!} + y_1y_3\right)f''(y_0) + \left(\frac{y_1^2y_2}{2!}\right)f'''(y_0) + \frac{1}{4!}\,y_1^4 f''''(y_0)$$
$$\vdots = \quad \vdots. \qquad \textbf{(2.9)}$$

Using (2.6) – (2.9) and linearity of the Laplace transform , we rewrite (2.5) as

$$\pounds\left(\sum_{n=0}^{\infty} y_n\right) = \frac{\alpha}{s} + \frac{1}{s}\pounds\big(R(x)\big) + \frac{1}{s}\,\pounds(Q(x))\sum_{n=0}^{\infty} y_n) + \frac{1}{s}\pounds(P(x)\sum_{n=0}^{\infty} A_n)\,.$$

The components, $y_n(x)$ of the solution $y(x)$ can easily determined by the following recurrence relations

$$£(y_0) = \frac{\alpha}{s} + \frac{1}{s}\,£\big(R(x)\big)\,, \qquad \textbf{(2.10)}$$

$$£(y_{n+1}) = \frac{1}{s}£(Q(x)y_n) + \frac{1}{s}\,£(P(x)A_n), \qquad n \geq 0 \qquad \textbf{(2.11)}$$

or

$$y_0 = £^{-1}\left[\frac{\alpha}{s} + \frac{1}{s}\,£\,(R(x))\right], \qquad \textbf{(2.12)}$$

$$\boldsymbol{y_{n+1}} = £^{-1}\left[\frac{\mathbf{1}}{\boldsymbol{s}}£\,(Q(x)y_n + \frac{1}{s}£(P(x)A_n\right], \qquad \boldsymbol{n \geq 0} \qquad \textbf{(2.13)}$$

Using (2.9)-(2.13) , we can easily find $y_n(x)$, $n \geq 0$ **.** We define n-terms approximation to the solution $y(x)$, by $\varphi_n(y) = \sum_{i=0}^{n-1} y_i$, and the solution can be obtained as the limit of the nth approximate solution , i.e.

$$\lim_{n\to\infty} \varphi_n(y) = y.$$

The Padé Approximates :

Here we investigate the construction of the Pade approximates for the functions studied . The main importance of the Padé approximation over the Taylor series approximation is that the Taylor series approximation can exhibit oscillations which may produces an approximation error bounds. Moreover, the Taylor series approximations can never blow- up in a finite region . To overcome these demerits we use the Padé approximations. The Padé approximations of a function is given by the ratio of two polynomials . The coefficients of the polynomial in both the numerator and the denominator are determined by using the coefficient in the Taylor series expansion of the function Padé approximation of a function , symbolized by [m/n], is rational function defined by

$$[m/n] = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_mx^m}{1 + b_1x + b_2x^2 + \cdots + b_nx^n} \qquad \textbf{(2.14)}$$

Where we considered $b_0 = 1$, and numerator and the denominator have no common factors.

In the LA-PA method we use method of the Padé approximation to the solution obtained by the Laplace decomposition method. This after treatment method improves accuracy of the proposed method.

3 Application

The results described in Section 2, are applied to some special cases of a class of nonlinear initial-value problems given in (1.1)-(1.2):

Example 1: Consider the following nonlinear Riccati differential equation

$$y' = y - 2y^2, \qquad \textbf{(3.15)}$$

$$y(0) = 1\,, \qquad \textbf{(3.16)}$$

With the exact solution $y(x) = \frac{1}{2-e^{-x}}$.

Solution: By applying the Laplace transform to both the sides of (3.15) and using the linearity property of the Laplace operator, we get

$s£[y] - y(0) = £[y] - 2\,£[y^2].$ **(3.17)**

Simplifying and using the initial condition (3.16) , we have

$£[y] = \frac{1}{s} + \frac{1}{s}\,£[y] - \frac{2}{s}£[y^2].$ **(3.18)**

Following the steps presented in Section 2 , assume the following

$$y^2 = \sum_{n=0}^{\infty} A_n,$$

(3.19)

Where

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\,[\textstyle\sum_{i=0}^{\infty} \lambda^i\, y_i]\right]_{\lambda=0}$$ **(3.20)**

Using (2.9) , the first few Adomian polynomials A_n for $N(y) = y^2$ are given by

$A_0 = y_0^2\,,$
$A_1 = 2y_0y_1,$
$A_2 = y_1^2 + 2y_0y_2\,,$
$A_3 = 2y_0y_3 + 2y_1y_2\,,$ **(3.21)**
$A_4 = 2y_0y_4 + 2y_1y_3 + y_2^2\,,$
$A_5 = 2y_0y_5 + 2y_1y_4 + 2y_2y_3\,,$

And by (2.6), solution $y(x)$ is given as

$y(x) = \sum_{n=o}^{\infty} y_n(x),$ **(3.22)**

Using refado,ser 1 given above rewrite (3.18) as

$$£\left[\sum_{n=0}^{\infty} y_n\,(x)\right] = \frac{1}{s} + \frac{1}{s}£[y] - \frac{2}{s}\,£\left[\sum_{n=0}^{\infty} A_n\right],$$

(3.23)

Now we define an iterative scheme

$£\,[y_0(x)] = \frac{1}{s}\,,$ **(3.24)**

$£[y_{n+1}(x)] = \frac{1}{s}\,£[y] - \frac{2}{s}\,£[\sum_{n=0}^{\infty} A_n]$ **(3.25)**

or

$y_0(x) = £^{-1}\left[\frac{1}{s}\right],$ **(3.26)**

$\boldsymbol{y_{n+1}(x)} = £^{-1}\left[\frac{1}{s}£[y]\right] - £^{-1}\left[\frac{2}{s}\,£[\sum_{n=0}^{\infty} A_n]\right]$ **(3.27)**

$\boldsymbol{y_0 = -x\,,\, y_1 = \frac{3x^2}{2}\,,\, y_2 = -\,\frac{13x^3}{6}\,,\, y_2 = \frac{25\,x^4}{8}}\,, \dots..$ **(3.28)**

Finally ,The approximate solution in series form is given by

$y(x) = -x + \frac{3x^2}{2} - \frac{13x^3}{6} + \frac{25x^4}{8} - \dots\dots\dots..$ **(3.29)**

To improve the accuracy of the approximate series solution we calculate the [2/2] Padé approximation of the series solution obtained from the Laplace decomposition method , which is given as follows :

$$y(x) \quad \frac{1+\frac{x}{2}+\frac{x^2}{12}}{1+\frac{3x}{2}+\frac{x^2}{12}} \tag{3.30}$$

h	Exact	Euler	Taylor	Adomian	Laplace-Pade
0.1	9.1311e-001	9.0000e-001	9.1311e-001	9.1310e-001	9.1311e-001
0.2	8.4655e-001	8.2800e-001	8.4639e-001	8.4622e-001	8.4655e-001
0.3	7.9417e-001	7.7368e-001	7.9215e-001	7.9086e-001	7.9417e-001
0.4	7.5206e-001	7.3133e-001	7.4063e-001	7.3517e-001	7.5207e-001
0.5	7.1763e-001	6.9750e-001	6.7526e-001	6.5859e-001	7.1765e-001

Table 1: Approximate solutions obtained by di erent numerical methods

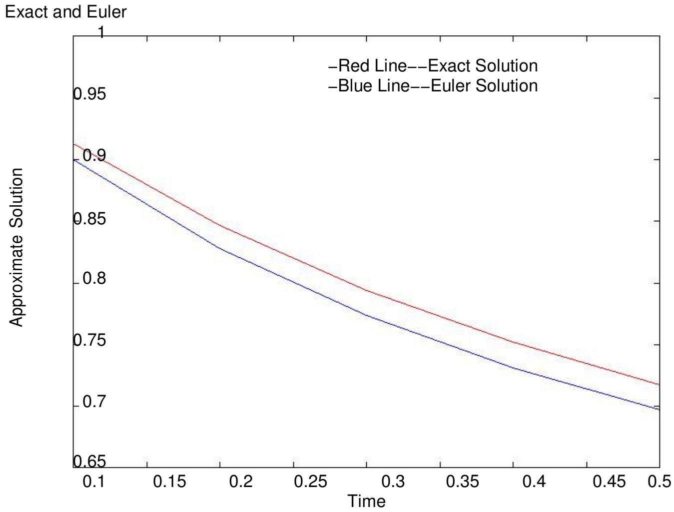

Fig. 1. Comparison between Exact and Euler method solution

Comparioson between Exact and Taylor

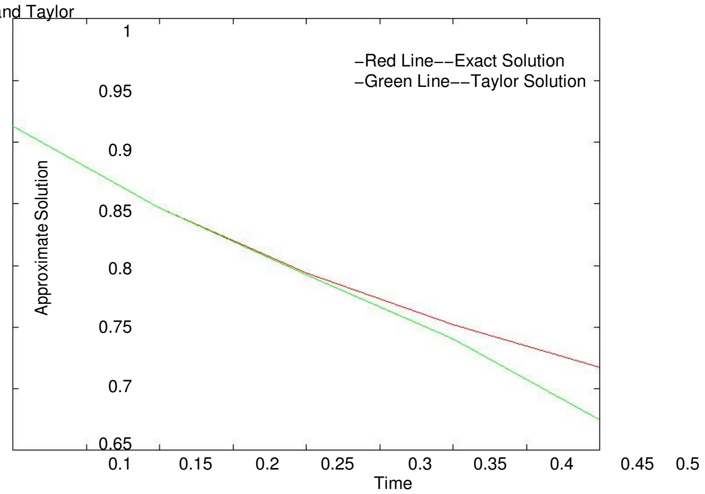

Fig. 2. Comparison between Exact and Taylor method solution

Comparioson between Exact and Adomian Series Solution

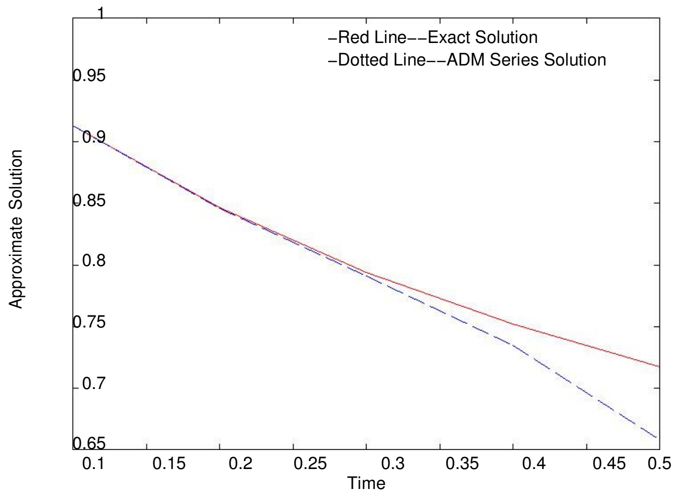

Fig. 3. Comparison between Exact and ADM series solution

Comparioson between Exact and LDM−Pade series Solution

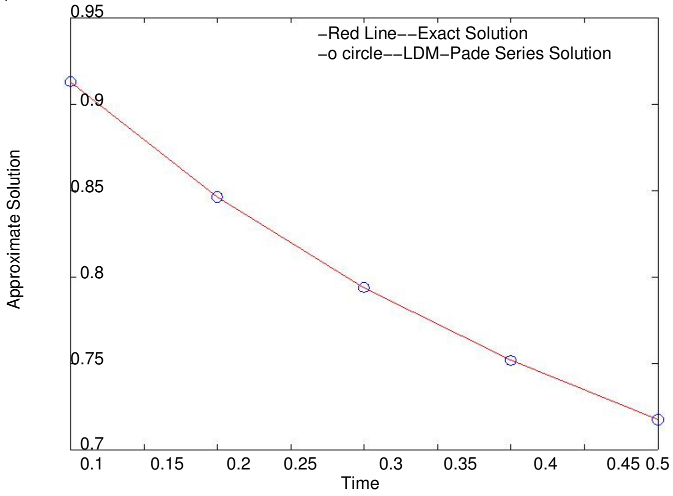

Fig. 4. Comparison

Comparioson between different solutions

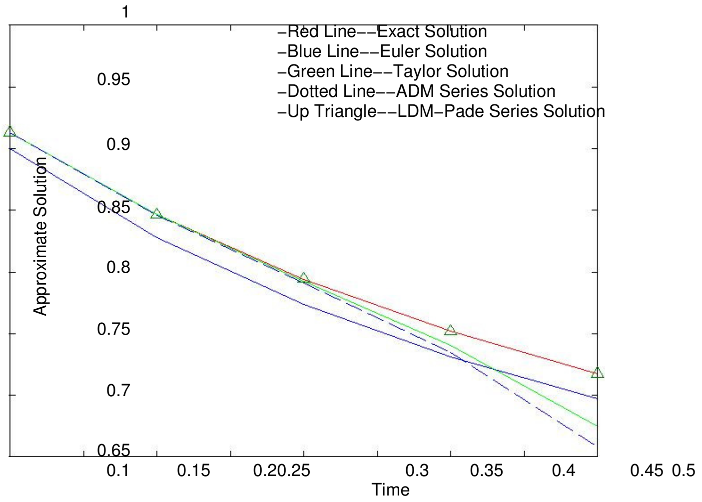

Fig. 5. Comparison between Exact and LDM-Pade solution

Table 2 Errors in the approximate solution obtained by di erent numerical methods

Step Size (h)	Euler [1]	Taylor [5]	Adomian [1]	LDM-Pade [2/2]
0.1	-1.3106e-002	-3.5079e-007	-5.6841e-006	1.0484e-008
0.2	-1.8547e-002	-1.5239e-004	-3.2305e-004	2.6139e-007
0.3	-2.0484e-002	-2.0133e-003	-3.3093e-003	1.5854e-006
0.4	-2.0727e-002	-1.1431e-002	-1.6893e-002	5.4435e-006
0.5	-2.0135e-002	-4.2373e-002	-5.9040e-002	1.3760e-005

Table (1) compares the results of the approximation using only four iterations of the Laplace decomposition and the Adomian decomposition method and with the exact solution. This table shows that the improvements of the Laplace decomposition-Pade method over the Euler method [1],the Taylor method [5] and the recently popular the Adomian decomposition method [1].

Table (2) shows the errors in the solution obtained by exact and other numer-ical methods and demonstrates the importance of the proposed method.

Conclude Remarks: In the present paper, we have presented a new method which we name as Laplace-Pade method for the solution of a nonlinear dif-ferential equation. This method is simple and highly accurate and stable in comparison with known classical numerical methods.Fourther applications of the method presented here will be combined in an subsequent analysis.

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